Revisiting fundamental concepts in Mediterranean predictability: Liouville equation and Tailored breds

Alejandro Hermoso and Víctor Homar

Meteorology Group, Physics department

Universitat de les Illes Balears, Palma, Mallorca, Spain

alejandro.hermoso@uib.es

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Introduction

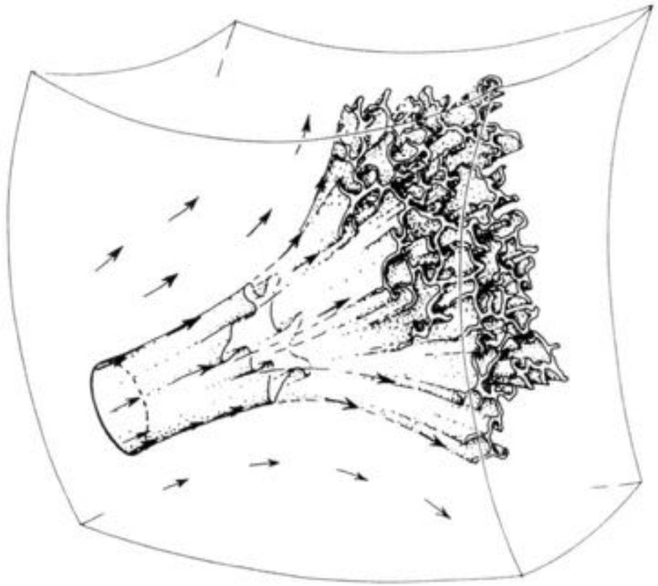
- Two sources of **uncertainty** in weather prediction: **initial state** and **physical processes**

- The mathematical framework to model uncertainty is **probability theory**

- **Uncertainty** is modelled through random variables **described by PDFs**. For multidimensional systems: multidimensional PDFs, representable over phase space

Introduction

- Most characteristics of **atmospheric PDFs** are largely **unknown.** Certain topologies would challenge common interpretations of EPS.



Penrose, 1989

- The **perfect-model evolution** of a system described by a state vector of random variables fulfils the **Liouville equation**

Objectives

- The purpose of this work is to explore some basic characteristics of simple solutions of the Liouville equation for low complexity systems
- Topological characteristics of the solutions will be analysed in detail
- Test the potential of LBVs to more efficiently initialize mesoscale EPS
- Investigate options to increase ensemble diversity and obtain a seamless scale representation compared to traditional BV

PART I: Liouville equation

General solution of the Liouville equation

- Liouville equation for the probability density function ρ given a dynamical system $\dot{\boldsymbol{X}} = \Phi(\boldsymbol{X})$:

$$\frac{\partial \rho(\boldsymbol{X},t)}{\partial t} + \sum_{k=1}^{N} \Phi_k(\boldsymbol{X},t) \frac{\partial \rho(\boldsymbol{X},t)}{\partial X_k} = -\chi(\boldsymbol{X},t)\rho(\boldsymbol{X},t) \qquad \chi = \sum_{k=1}^{N} \frac{\partial \Phi_k}{\partial X_k}$$

- If the **method of characteristics** can be used in the problem at stake, then the general solution is:

$$\rho(\boldsymbol{X},t) = \rho_0(\boldsymbol{\alpha}) exp\Big(- \int_0^t \chi(\boldsymbol{X}(\boldsymbol{\alpha},t'),t')dt' \Big)$$

$\boldsymbol{\alpha}$ is the state vector at t = 0

Application to a Low dimensional barotropic model

- From barotropic vorticity equation under beta-plane approximation is:

$$\frac{\partial}{\partial t}\nabla^2\psi = \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} - \beta\frac{\partial\psi}{\partial x}$$

- A solution consisting in a highly truncated Fourier series expansion for $\nabla^2\psi$ with three time dependent amplitudes (X_1, X_2 and X_3) and phase angles linearly dependent on time (θ_2 and θ_3) is proposed (Peagle and Robl, 1977):

$$\nabla^2\psi = X_1(t)cos(ly) + X_2(t)cos(kx+\theta_2) + 2X_3(t)sin(kx+\theta_3)sin(ly)$$

$$\theta_2 = \frac{\beta}{k}t \qquad \theta_3 = \frac{\beta k}{k^2+l^2}t$$

Analytical solution of the system

$$X_1 = X_1^* dn(h\tau + \phi, k_0^2)$$

$$X_2 = X_2^* sn(h\tau + \phi, k_0^2)$$

$$X_3 = X_3^* cn(h\tau + \phi, k_0^2)$$

$$\tau = \frac{sin(\gamma t)}{\gamma}$$

dn, sn and cn are the Jacobi elliptic functions

If β (and γ) are 0, τ is t

$$k_0^2 = \frac{1 - \frac{C_2}{C_3}\frac{X_{30}^2}{X_{20}^2}}{1 - \frac{C_2}{C_1}\frac{X_{10}^2}{X_{20}^2}}$$

$$\phi = sn^{-1}\Big[\frac{1}{\sqrt{1 - \frac{C_2}{C_3}\frac{X_{30}^2}{X_{20}^2}}}, k_0^2\Big]$$

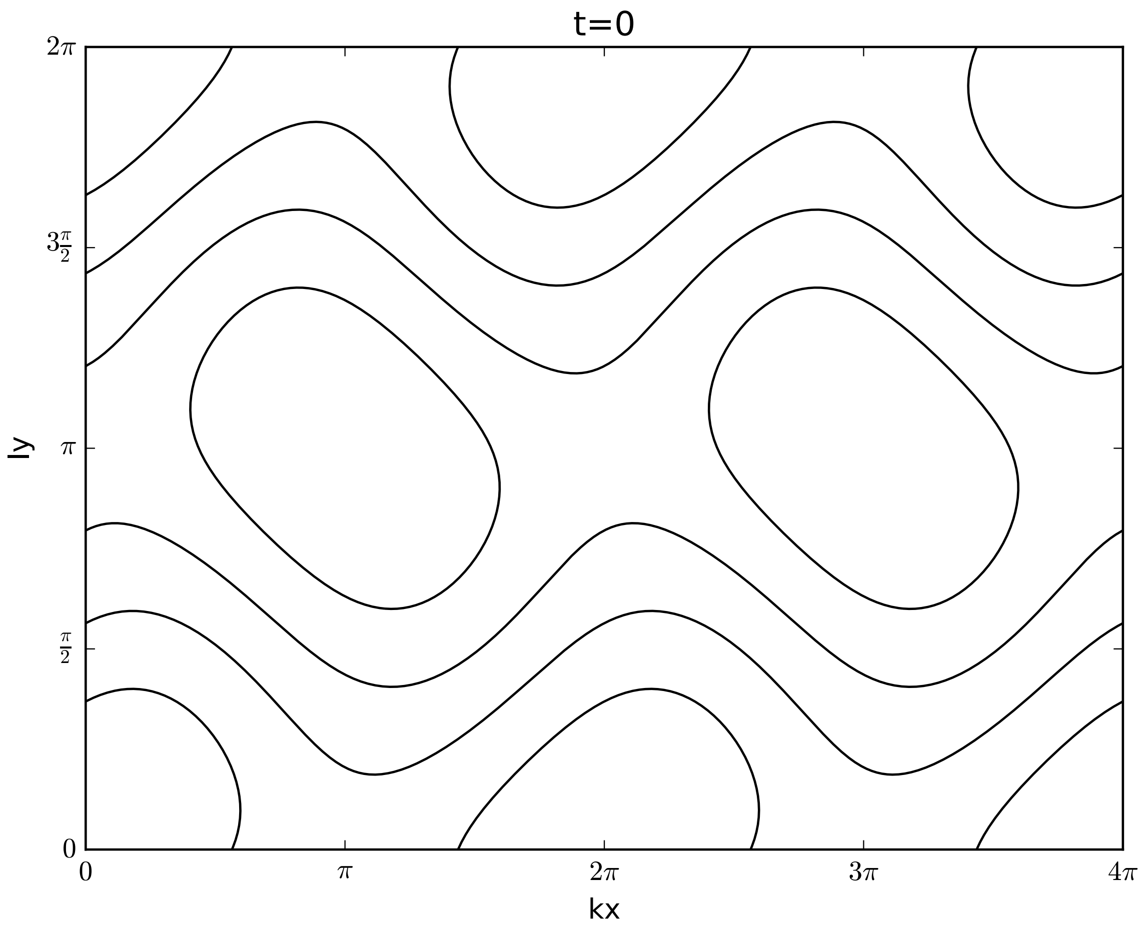
$$X_1^* = \frac{X_{10}}{dn(\phi, k_0^2)}$$

$$X_2^* = \frac{X_{20}}{sn(\phi, k_0^2)}$$

$$X_3^* = \frac{X_{30}}{cn(\phi, k_0^2)}$$

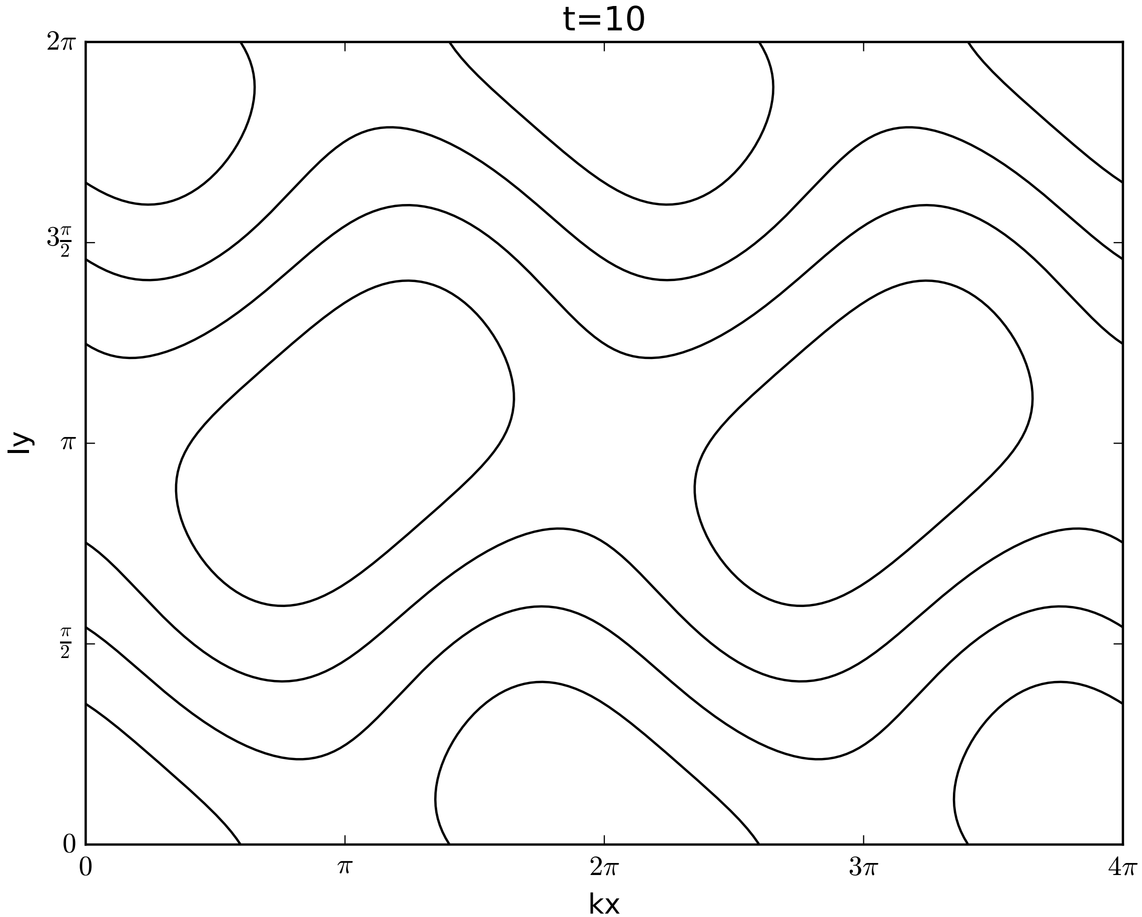
$$h = C_2 \frac{X_1^* X_3^*}{X_2^*}$$

Analytical solution of the system



$X_{10} = 0.12$
$X_{20} = 0.24$
$X_{30} = 0.10$
$\alpha = 2$
$\beta = 0$

Analytical solution of the system



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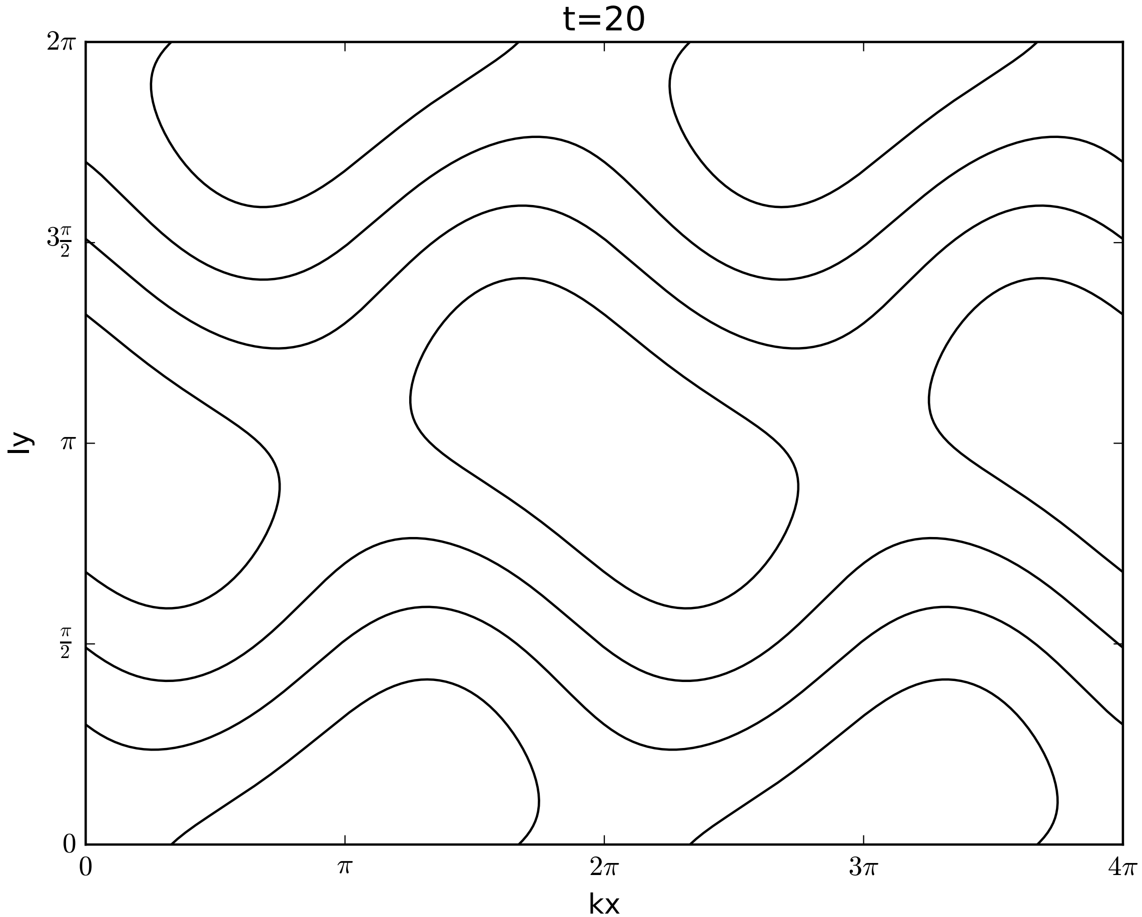

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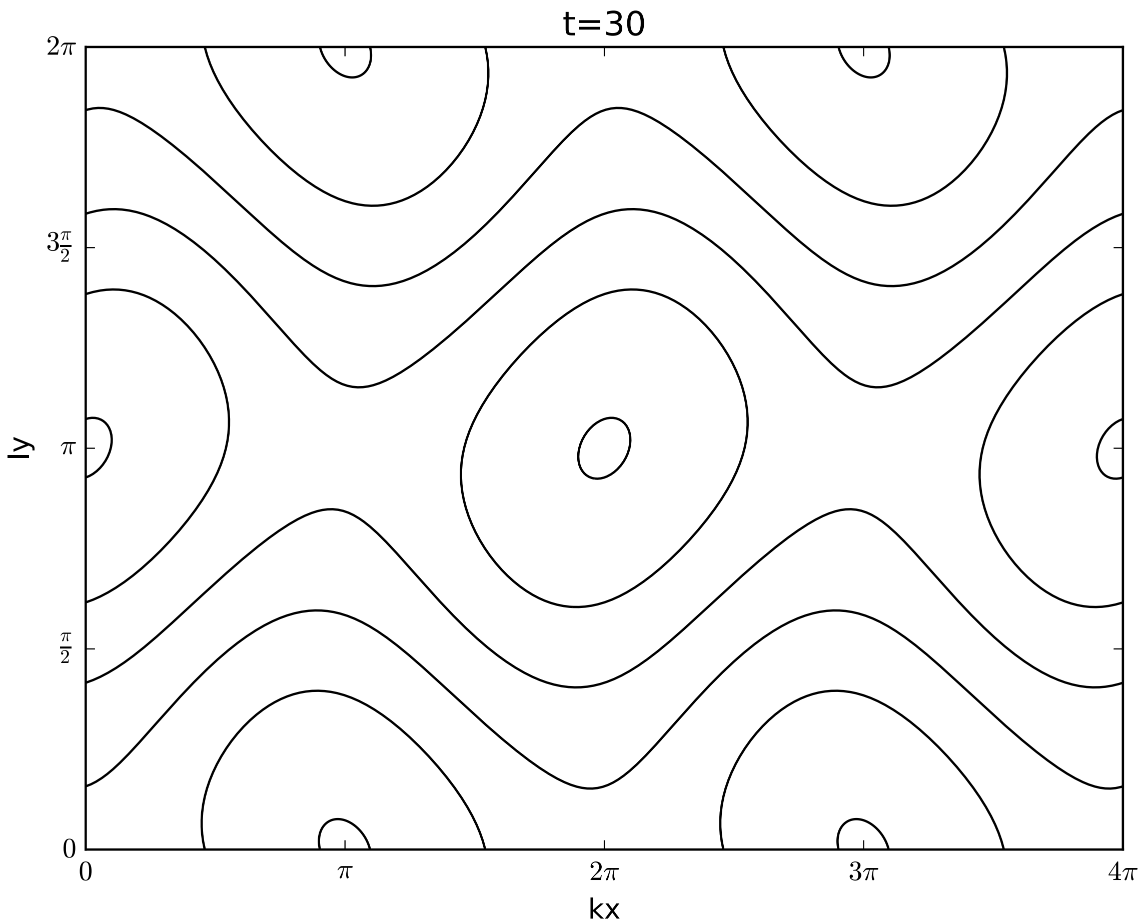

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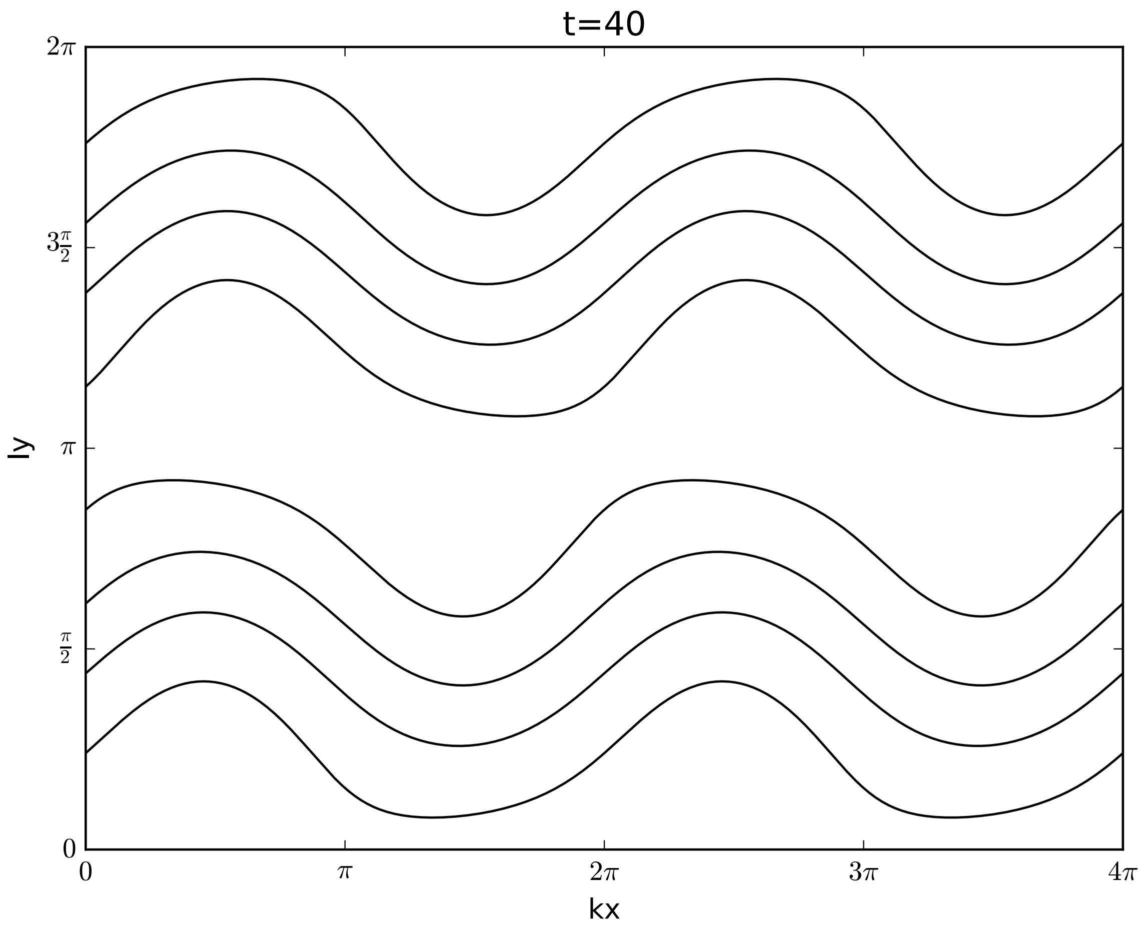

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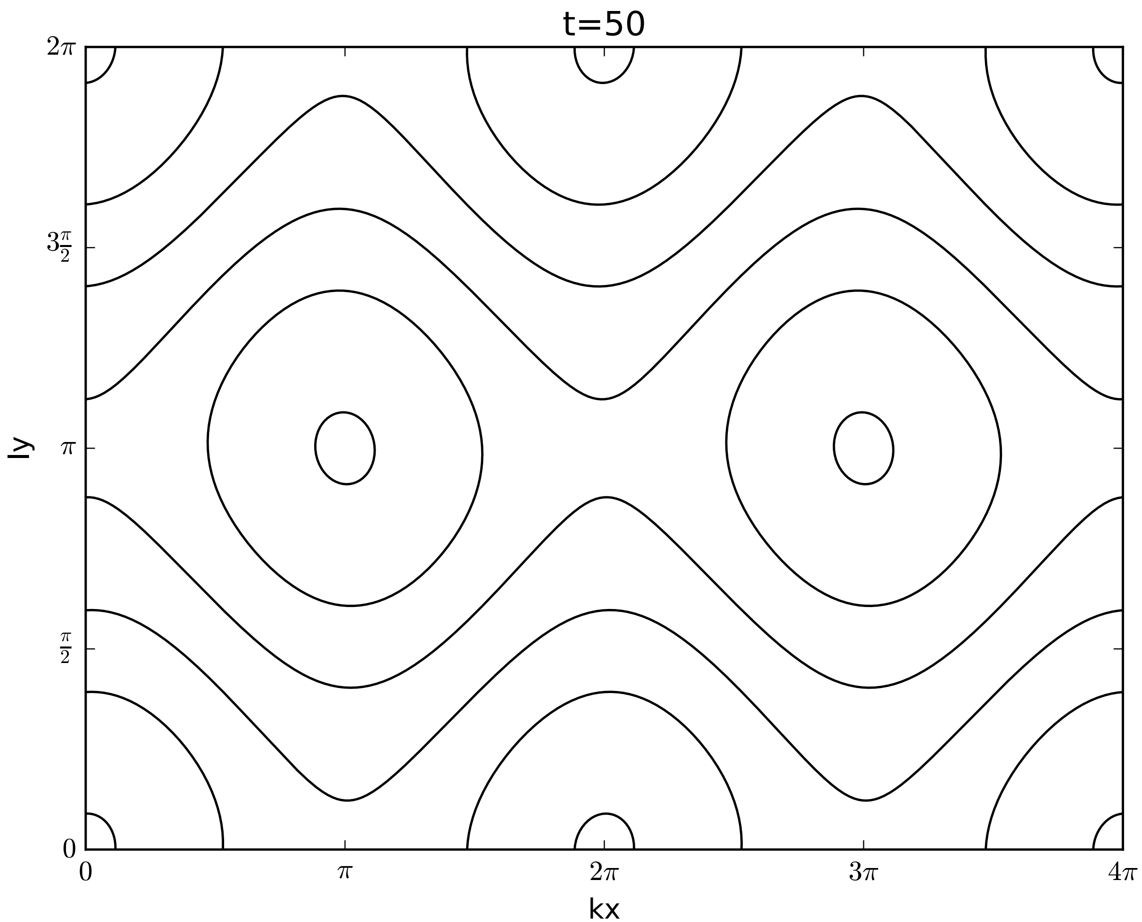

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$\alpha = 2$
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$$\rho(\boldsymbol{X},t) = \rho_0(\boldsymbol{\alpha}) exp\Big(-\int_0^t \chi(\boldsymbol{X}(\boldsymbol{\alpha},t'),t')dt'\Big)$$

Solution of the Liouville equation

- Applying the general solution to the system obtained from the barotropic model yields:

$$\rho(X_1,X_2,X_3,t) = \rho_0(X_{10}(X_1,X_2,X_3,t),X_{20}(X_1,X_2,X_3,t),X_{30}(X_1,X_2,X_3,t))$$

Solution of the Liouville equation

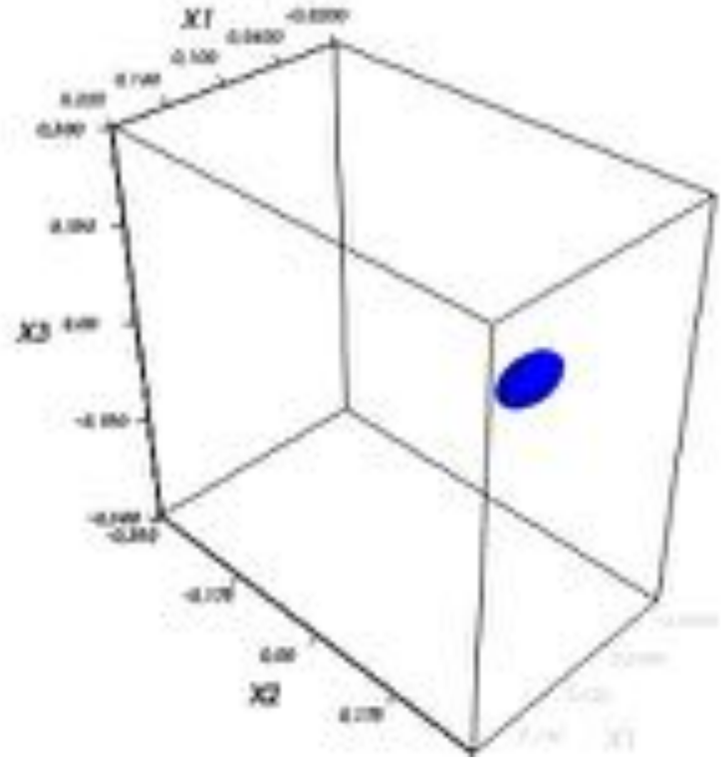
- The initial ρ is defined here as a three dimensional Gaussian distribution:

$$\rho_0(X_{10}, X_{20}, X_{30}) = k\ exp\Big[-\Big(\frac{(X_{10}-\mu_1)^2}{2\sigma^2}+\frac{(X_{20}-\mu_2)^2}{2\sigma^2}+\frac{(X_{30}-\mu_3)^2}{2\sigma^2}\Big)\Big]$$

k: normalization constant
μ_k: means in each direction
σ: standard deviation

- μ_1 = 0.12, μ_2 = 0.24, μ_3 = 0.1, σ = 0.01

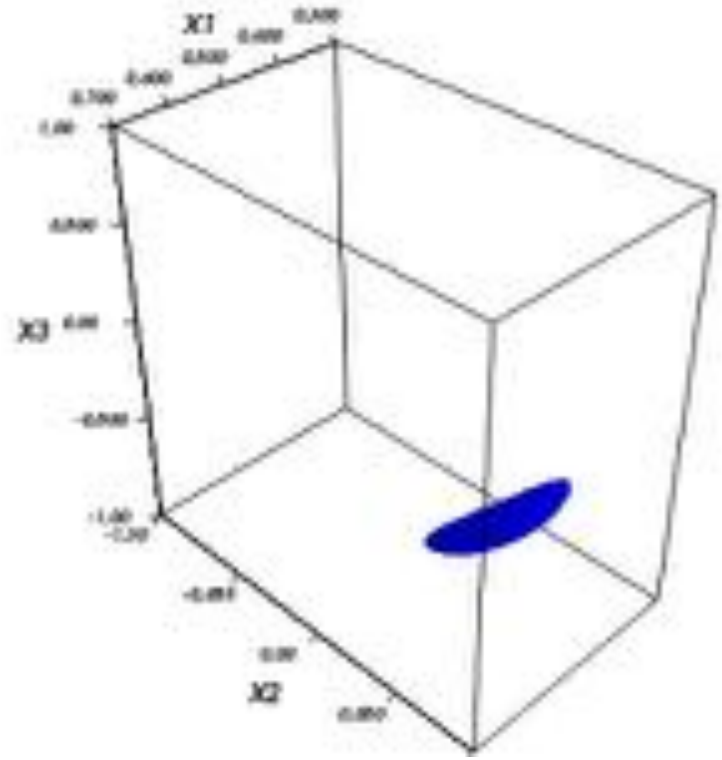
Evoloution of ρ_1 (α = 2, β = 0)



t = 0

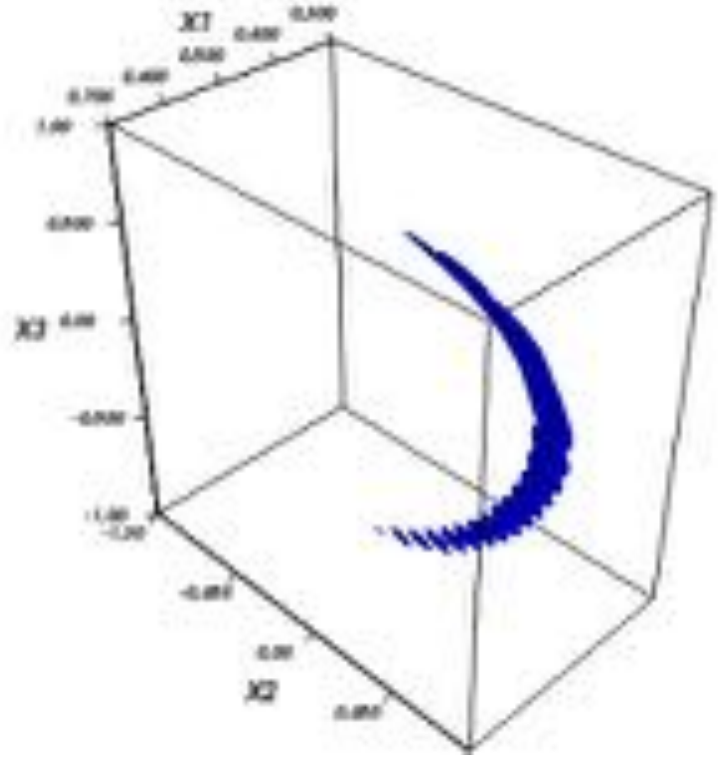
$$\alpha = \frac{k}{l}$$

Evoloution of ρ_1 ($\alpha = 2$, $\beta = 0$)



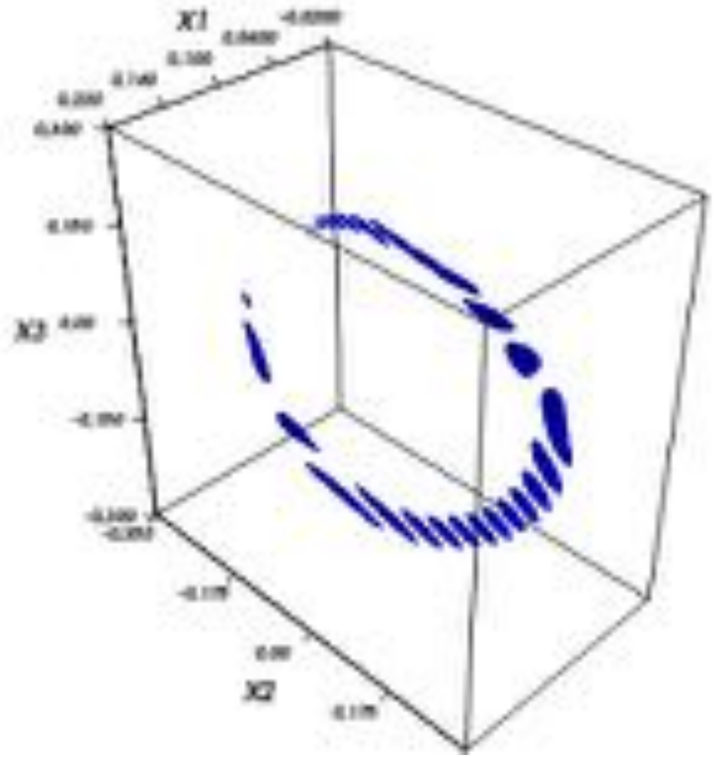
t = 10

Evoloution of ρ_1 ($\alpha = 2$, $\beta = 0$)



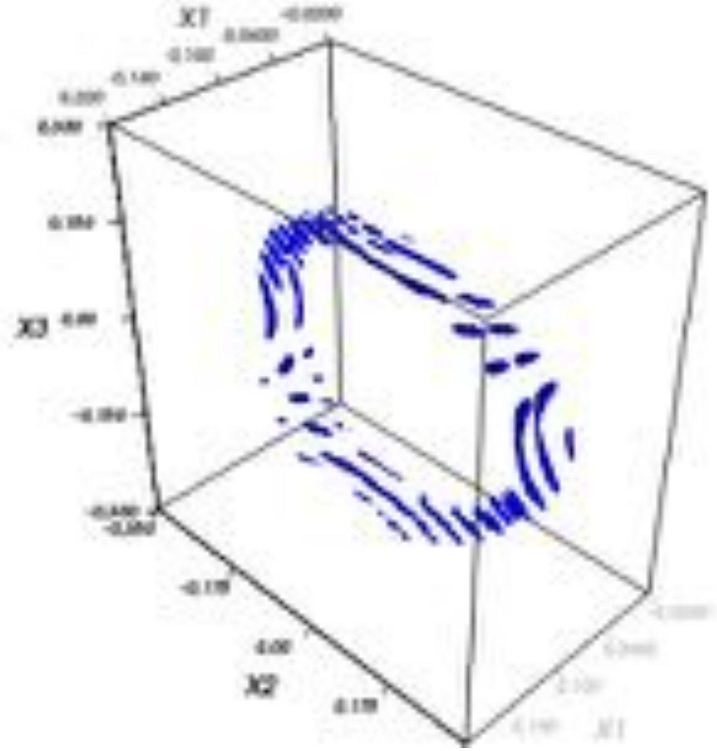
t = 50

Evoloution of ρ_1 (α = 2, β = 0)



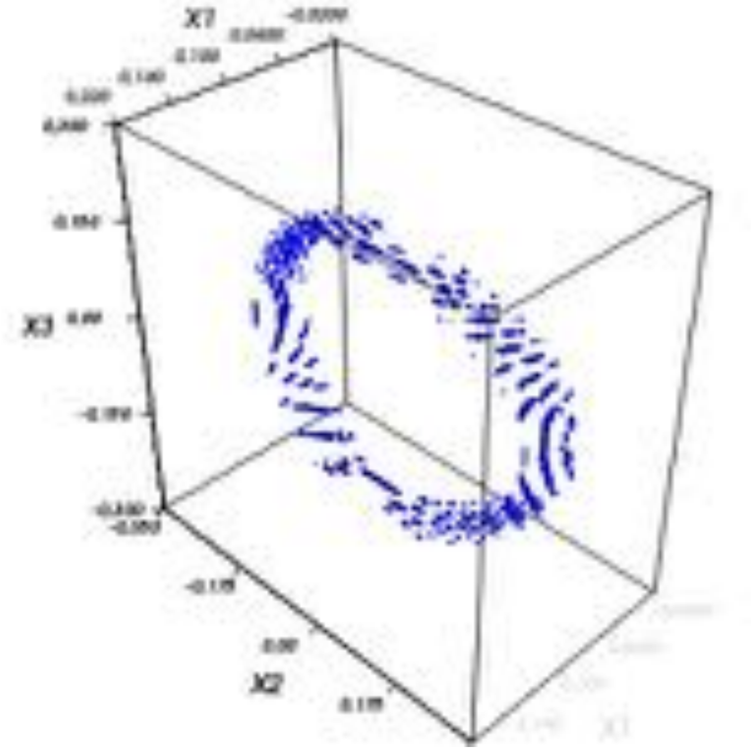
t = 100

Evoloution of ρ_1 (α = 2, β = 0)



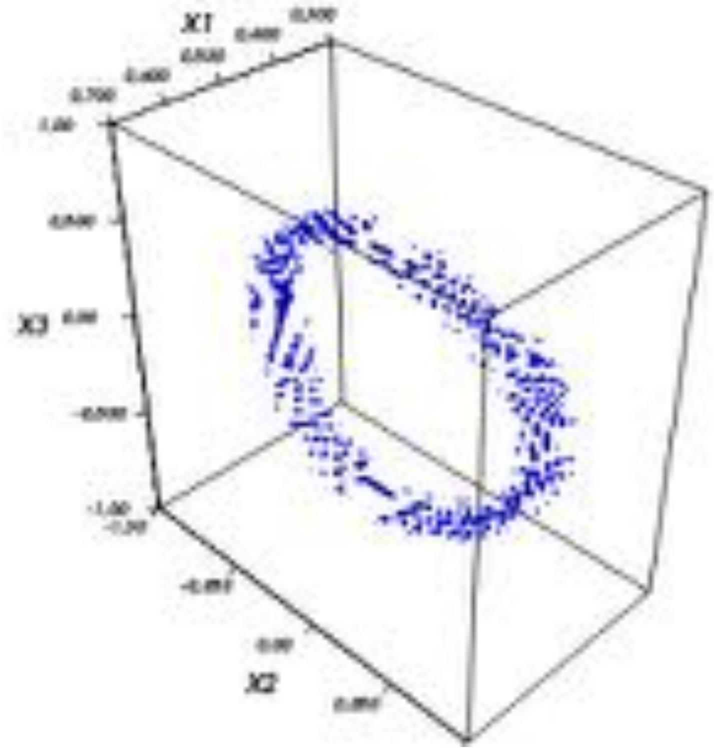
t = 250

Evoloution of ρ_1 (α = 2, β = 0)



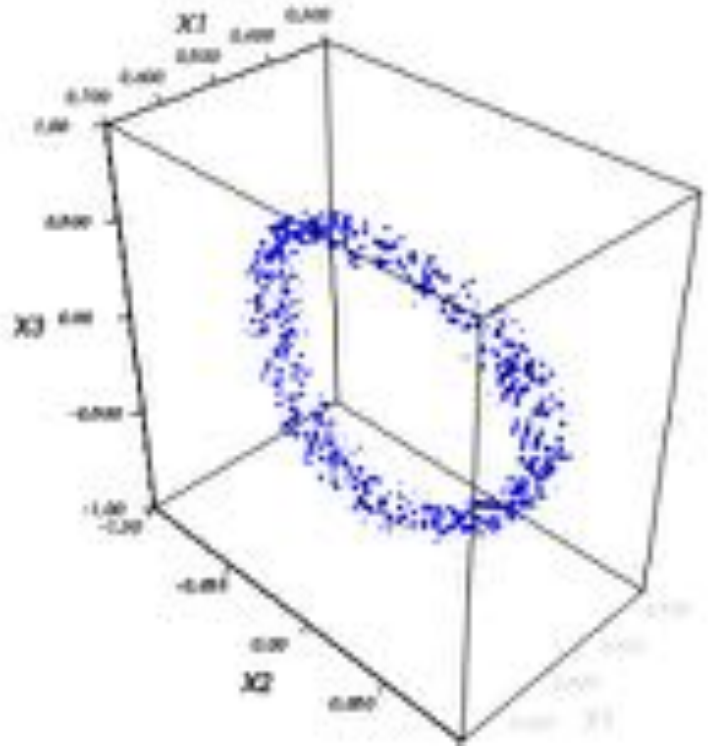
t = 500

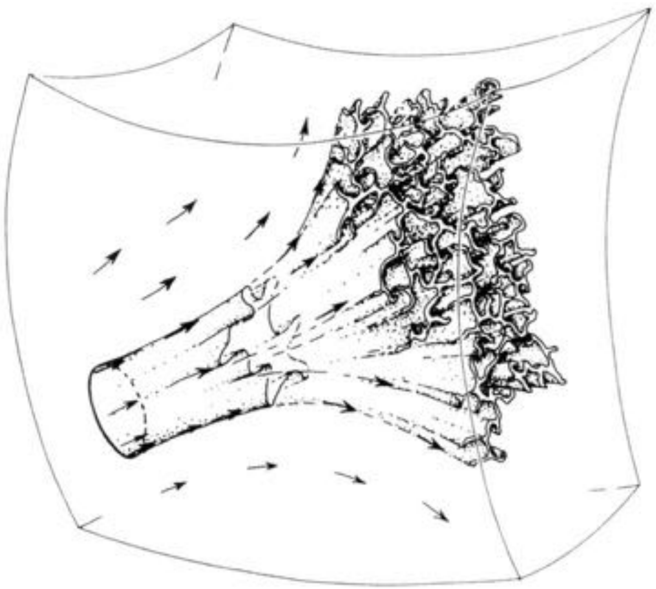
Evoloution of ρ_1 ($\alpha = 2$, $\beta = 0$)



t = 1000

Evoloution of ρ_1 ($\alpha = 2$, $\beta = 0$)





Numerical solution of the Liouville equation for a barotropic model

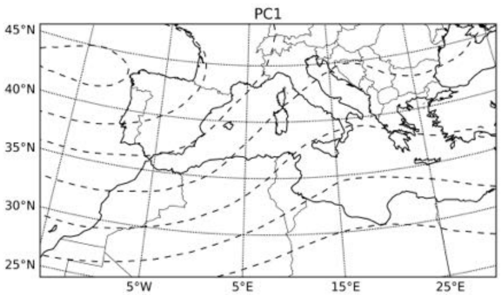
- Solution of the Liouville equation for a full barotropic model in a 91x51 grid with a brute-force approach

- Total number of model integrations ${N_P}^{N_D}$ $O(10^{7000})$

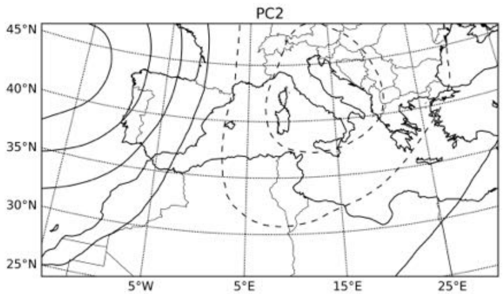
- Principal Component Analysis is applied to reduce system dimensionality using ECMWF analysis data

- 5 PC are kept

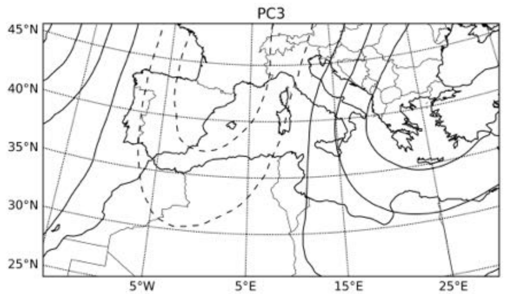
Numerical solution of the Liouville equation for a barotropic model



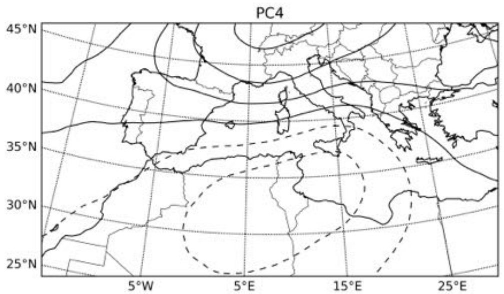
PC1



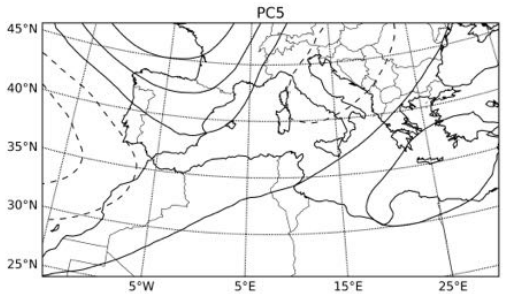
PC2



PC3



PC4



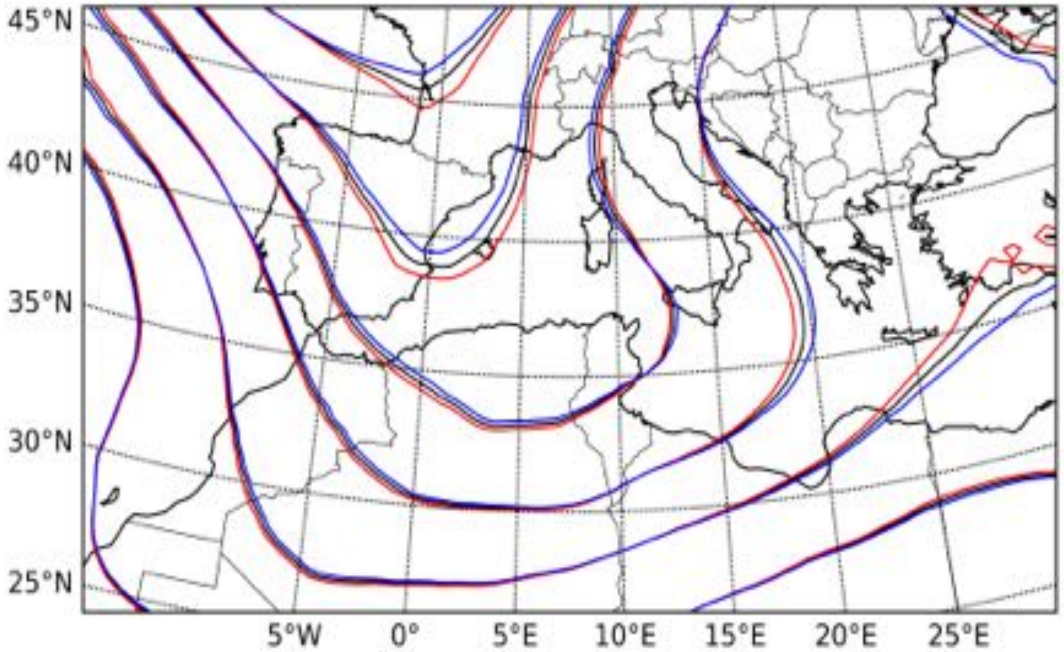
PC5

Numerical solution of the Liouville equation for a barotropic model

- New phase space is 5-dimensional (5 PC retained)
- It is discretised in $N_P = 75$ points in each dimension
- Initial PDF is a gaussian distribution
- Mean is obtained from the ERA5 500 hPa geopotential on 7th November 2014
- Liouville equation is solved with a lead time of 24h

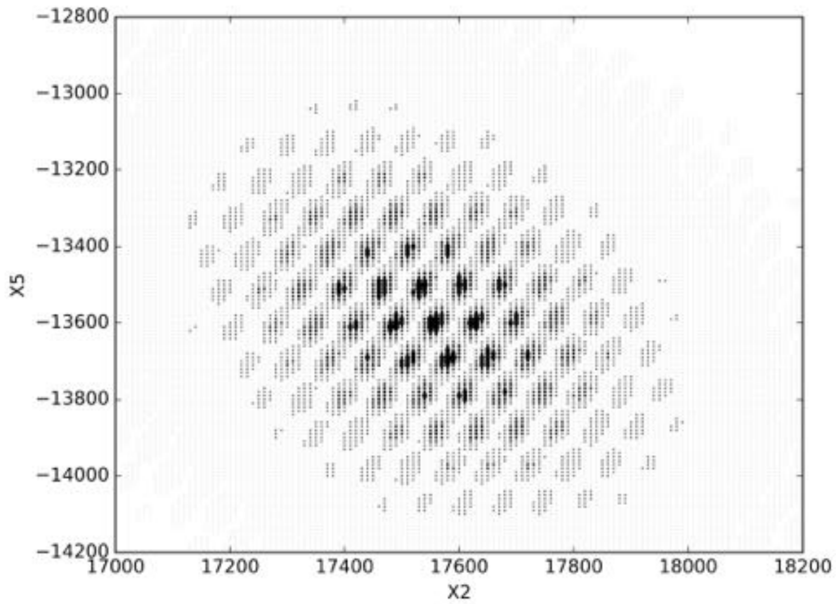
Numerical solution of the Liouville equation for a barotropic model

- Initial conditions

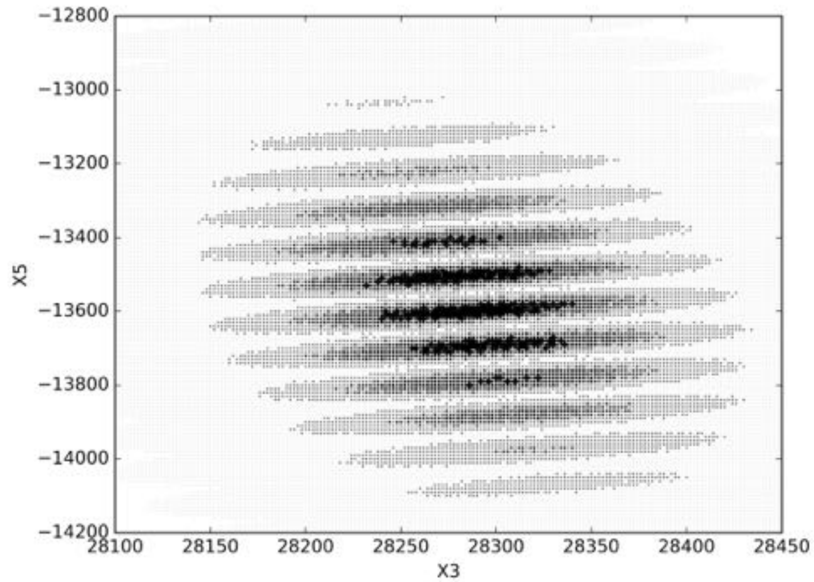


Mean + and - 2σ in X3 for ρ_{bar1}

Results



X2-X5 plane



X3-X5 plane

Part I conclusions

- The **Liouville equation** has been **solved analytically** for a low complexity system
- The **granularity** of the analytical solution for certain values of the parameters reveals a **serious predictability challenge**
- A **granular solution** is also identified in a more **realistic model**
- These results **challenge** most current **ensemble prediction** products that are based on compact PDF

PART II: Tailored Bred Vectors

Motivation and objectives

- One of the main problems of EPS is the underdispersion
- Extreme events may not be represented
- In order to improve the high resolution short-range forecast of extreme events, ensemble **spread** must be **controlled** (typically increased)

New perturbation method: Tailored Bred vectors

- This method enables to **increase the ensemble size at no** bred generation **cost**

- Allows a **seamless scale representation** unlinking scales of forecast interest from bred generation strategy (and so recycling period)

- BVTEP are generated by combining different BV with different scales and amplitudes:

$$P = \sum_i \gamma_i \delta x_i^{1/\beta_i}$$

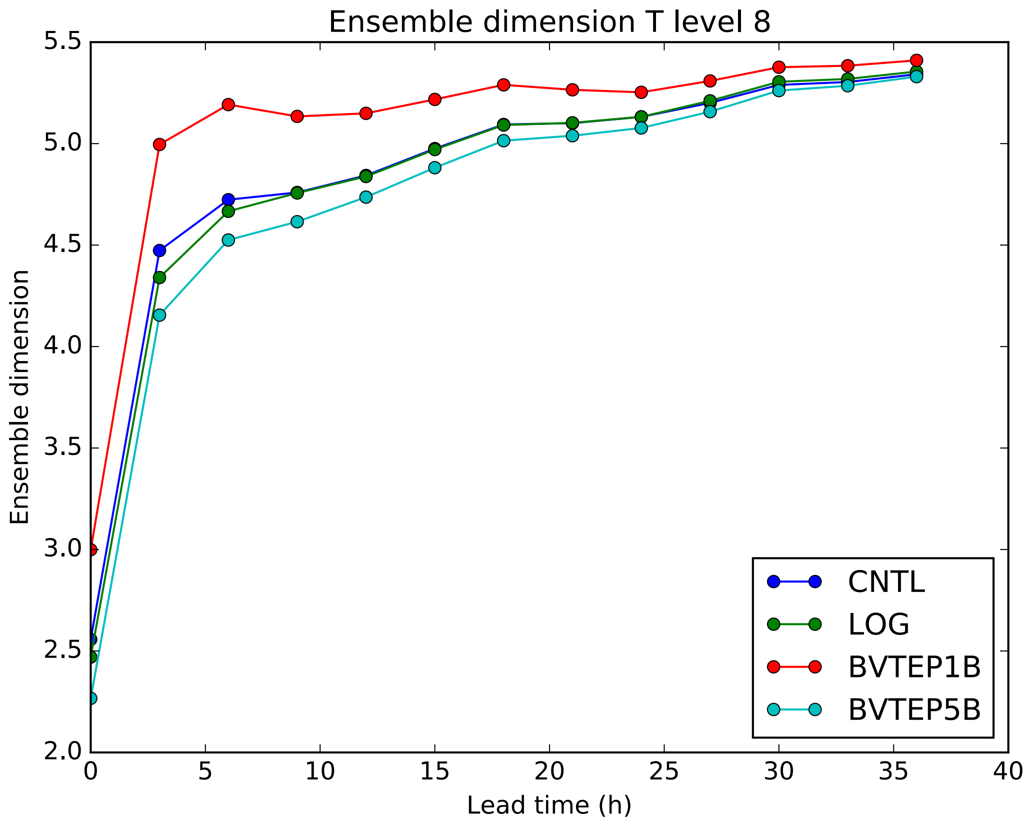
i: 1,…n breds

Ensemble configurations

- **CNTL**: 5 ABV perturbations (1 bred per perturbation)
- **LOG**: 5 LBV perturbations (1 bred per perturbation)
- **BVTEP1B**: 5 ABV (1 ω-rescaled bred per perturbation)
- **BVTEP5B**: 5 ABV (Linear combination of up to 5 ω-rescaled bred per perturbation)

Results

Variation of **ensemble dimension** with lead times for T at a model level close to 850 hPa



Part II conclusions

- The **ensemble diversity** is **similar** for forecasts perturbed with **arithmetic** and **logarithmic** rescaled **bred vectors**

- Modifying the **scale** of the initial perturbations **increases ensemble diversity and skill**

- **Puzzling result**: better skill from particular case of general method…

- The methodology should be tested in a severe weather event

Acknowlegments

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"A way to make Europe"

ENSEMBLE DIMENSION

- **Ensemble dimension** is computed from the eigenvalues μ_i of the covariance $\boldsymbol{C}_{ij}$ matrix of a set of states:

- Perturbations covariance matrix:

$$\boldsymbol{C}_{ij} = \frac{\langle \boldsymbol{b}^{(i)}, \boldsymbol{b}^{(j)} \rangle}{L||\boldsymbol{b}^{(i)}||_2||\boldsymbol{b}^{(j)}||_2} \quad \langle \boldsymbol{b}^{(i)}, \boldsymbol{b}^{(j)} \rangle = \sum \boldsymbol{b}^{(i)}(\boldsymbol{x})\boldsymbol{b}^{(j)}(\boldsymbol{x})$$

- Ensemble dimension

$$D = \frac{\left(\sum_{i=1}^{k} \sqrt{\mu_i}\right)^2}{\sum_{i=1}^{k} \mu_i}$$

D quantifies the number of independent vectors in the set (ensemble).